Tidal Deformability of Neutron Stars

Author: Pol Morell Ferrer
Facultat de Física, Universitat de Barcelona, Diagonal 645, 08028 Barcelona, Spain.

Tutor: Mario Centelles Aixalà

Abstract: A study on neutron stars, regarding the equations of state and the structure equations used to model them, has been performed as a benchmark for investigating the tidal deformability of these compact stars, which may be accessed through gravitational wave detections, and its contribution in obtaining generic (model-independent) relations between neutron star relevant magnitudes.

I. INTRODUCTION

Neutron stars (NS) are extremely dense objects, with masses comparable to that of the Sun, and radii of around 10 km. As described in, e.g. Refs. [1,2], at such densities, most of the interior of the star consists of a nuclear liquid composed by neutrons with a certain fraction of protons, electrons and muons that maintain the β-equilibrium. This constitutes the core of the NS, a region that accounts for over $\sim 90\%$ of its whole mass and size. Moving outside from the core, both density and pressure decrease. At a certain point, when the density is low enough, matter inhomogeneities start to prevail, and protons distribute in clusters that define a solid lattice (in order to minimize the Coulomb repulsion), which is surrounded by a gas of free neutrons and a background of electrons such that the whole system is charge neutral. This region of the star is called the inner crust. At even lower densities that reach up to the surface of the NS, in what is called the outer crust, neutrons are finally confined within the nuclear clusters and thus matter consists of a lattice of neutron-rich nuclei embedded in a degenerate electron gas. The whole NS crust is an external layer around ~ 1 km deep.

The physics of isospin-rich ultradense nuclear matter is yet to be fully understood, due to its complex nature and the technological impossibility to replicate it in the laboratory. As a consequence [3-5], astronomical observations on NS play an essential role in this field, as the only viable source of experimental data. Many of the magnitudes of astronomical interest (such masses, as radii and moments of inertia), which can be obtained indirectly from observational data, are strongly dependent on the nuclear equation of state (EoS). Therefore, great effort is directed towards learning how to use NS observational data to constrain the EoS.

Since the first gravitational wave (GW) detection coming from the coalescense of a binary system of NS measured by the LIGO-Virgo collaboration [12] (GW170817), this kind of observations has proven to carry much information about the merging objects themseves. From the GW signal, it is possible to extract the *chirp mass* (which is a weighted average of masses) of the system, that influences the inspiral frequency of the merging bodies; and also the tidal deformability of the system, which affects the signal by adding phase corrections to the point-mass dynamics, as described in Refs. [3-5]; and also, which is highly sensitive to the EoS.

In this work, we have started by studying the equation of state, particularly by analytically deducing and analysing the properties of a simplified EoS. Then, employing some of the EoSs commonly featured in the literature, we have solved the equations for the structure, the tidal deformability and the moment of inertia for a NS. We have used the resulting data to study the behaviour of these magnitudes and their dependence on the EoS. Finally, and motivated by the results found in the literature, we have deduced some universal (EoS independent) relations involving the tidal deformability, which are of great interest when coming to imposing restrictions on the EoS, while also providing helpful ways to constrain non-measurable parameters (such as, in GW observations, the radius of the star or its moment of inertia) from observable magnitudes (such as, again for GW detections, the tidal deformability).

II. THEORETICAL BACKGROUND

A. TOV Structure Equations

Following Refs. [6,7], for very compact stellar objects such as NSs, which require for the effects of general relativity to be introduced, imposing hydrostatic equilibrium under spherical symmetry leads to the Tolmann-Oppenheimer-Volkoff (TOV) equations, which read:

$$\frac{dp(r)}{dr} = -\frac{Gm(r)}{r^2}\frac{\varepsilon(r)}{c^2}\left[1+\frac{p(r)}{\varepsilon(r)}\right] \left[1+\frac{4\pi r^3 p(r)}{m(r)c^2}\right]\left[1-\frac{2Gm(r)}{rc^2}\right]^{-1}, \tag{1}$$

$$\frac{dm(r)}{dr} = 4\pi r^2\frac{\varepsilon(r)}{c^2}, \tag{2}$$

where *p(r)* and $\varepsilon(r)$ stand for the pressure and the energy density, respectively; *m(r)* for the mass contained in a sphere of radius r; G is the gravitational constant and c is the speed of light. These are foundational structure equations that lead to the energy density profile inside the neutron star by solving from the center ($r = 0$), with initial condition $m(0) = 0$ and an arbitrary central pressure, to the surface (which is reached when pressure vanishes). Solving for different central pressures defines the mass-radius (M-R) relation. Nonetheless, another equation is needed corresponding to the EoS −which relates the pressure and the energy density, $p(\varepsilon)$− in order to compute the stellar structure.

B. Equation of State

We have described the core of the neutron star, according to Refs. [1,2,7], as composed by a uniform $\beta-$stable nuclear liquid. As such, the EoSs used to describe it consists of two independent contributions: one coming from leptons (electrons and muons), and the other coming from neutrons and protons (the *nuclear EoS*). In this work we will not consider more exotic compositions, such as hyperons or quark matter.

Due to the extreme densities found in the core, the leptonic contribution can be considered as an ultrarelativistic free electron Fermi gas at $T = 0$, as the temperatures inside the core fall way below the Fermi temperature. Therefore, the energy density in terms of the electron number density (n_e) reads [7]:

$$\mathcal{H}_e(n_e) = \frac{m_e^4 c^5}{8\pi^2\hbar^3}\left[\sqrt{1+x^2}\left(2x^3+x\right) - \operatorname{arcsinh}(x)\right], \tag{3}$$

where $x \equiv (3\pi^2 n_e)^{1/3}\hbar/m_e c$, and m_e is the electron rest mass. In the ultrarelativistic limit, this expression can be simplified to read:

$$\mathcal{H}_e(n_e) = \frac{3}{4}(3\pi^2)^{1/3} n_e^{4/3}\hbar c, \tag{4}$$

Furthermore, for large electron densities muons may start appearing through the following equilibrium [1]:

$$\begin{aligned} e^- &\longrightarrow \mu^- + \nu_\mu + \bar{\nu_e} \\ \mu^- &\longrightarrow e^- + \bar{\nu_e} + \nu_\mu, \end{aligned} \tag{5}$$

which is implemented into the leptonic equation of state by adding a muon contribution analogous to Eq. (3), whenever the electron chemical potential ($\mu_i = \partial\mathcal{H}_i/\partial\rho_i$) grows higher than the muon rest mass. The equilibrium between muons and electrons ($\mu_\mu = \mu_e$) leads to:

$$n_\mu^{2/3} = n_e^{2/3} - \frac{(m_\mu^2 - m_e^2)c^2}{\hbar^2(3\pi^2)^{2/3}}, \tag{6}$$

if the muon number density (n_μ) satisfies $n_\mu \in \mathbb{R}^+$; otherwise, $n_\mu = 0$.

The hadronic contribution to the EoS, following Ref. [8], contains the kinetic energy of both protons and neutrons, and the interaction energy between them; the latter being dependent on the model chosen to describe nuclear interactions. In this work, we have described them using Skyrme parametrizations, which are effective descriptions based on contact potentials (e.g. $V(x_1, x_2) \sim \delta(x_1 - x_2)$). As a result, the energy density for *infinite* nuclear matter can be written in terms of powers of n_p and n_n (number densities for protons and neutrons). The following energy density, which corresponds to the simplified version that we have used as a preliminary study on the properties of the Skyrme EoS, may serve as an example:

$$\begin{aligned} \mathcal{H}_N(n_n, n_p) &= \frac{3\hbar^2}{10m_N}(3\pi^2)^{2/3}\left(n_n^{5/3} + n_p^{5/3}\right) \\ &+ \frac{t_0}{4}\left[(1+x_0)n^2 - (2x_0+1)(\rho_n^2 + n_p^2)\right] \\ &+ \frac{t_3}{24}n^\sigma\left[2n^2 - (n_n^2 + n_p^2)\right] + m_N c^2 n, \end{aligned} \tag{7}$$

where $m_N \approx m_p \approx m_n$, and all the other parameters ($t_0 = -1803.51$ MeV·fm^3; $t_3 = 12913.52$ MeV·fm^4; $x_0 = 0.16$; $\sigma = 1/3$) are semi-empirical and adjusted, depending on the particular Skyrme interaction, based on experimental results or matching certain constraints (e.g. imposing the saturation density to occur at $n = 0.16$ fm^{-3}, requiring an energy per particle of −16.0 MeV, etc.). The variables can be redefined as the asymmetry between neutrons and protons, $\delta = (n_n - \rho_p)/2$, and the total number density, $n = n_n + n_p$.

In addition, as the liquid core consists mainly of individual neutrons which are unstable and tend to decay via $\beta-$decay, the following equilibrium must be included in the EoS [1,7]:

$$\begin{aligned} n &\longrightarrow p + e^- + \bar{\nu_e} \\ p + e^- &\longrightarrow n + \nu_e. \end{aligned} \tag{8}$$

The condition of $\beta-$stability can be implemented via matching the chemical potentials of both sides of the equilibrium (and neglecting the chemical potential of neutrinos, as they travel through the star almost without interacting at all). The resulting equation, $\mu_n + \mu_p = \mu_e$, combined with the charge neutrality condition and Eq. (6), constitute a system that allows for δ and n_e to be determined for each value of n:

$$\hbar c(3\pi^2 n_e)^{1/3} = \frac{2}{n}\frac{\partial\mathcal{H}_N}{\partial\delta} \tag{9}$$

$$n_e + n_\mu = \frac{1}{2}n(1-\delta). \tag{10}$$

Finally, the equation of state itself, $p(\varepsilon)$, can be retrieved from:

$$\varepsilon(n) = \mathcal{H}_e(n) + \mathcal{H}_\mu(n) + \mathcal{H}_N(n), \tag{11}$$

$$p(n) = n_e\frac{\partial\mathcal{H}_e}{\partial n_e} + n_\mu\frac{\partial\mathcal{H}_\mu}{\partial n_\mu} + n\frac{\partial\mathcal{H}_N}{\partial n} - \varepsilon(n). \tag{12}$$

Aside from the core, in this work, the outer crust has been modelled with the Douchin-Haensel EoS, while the inner crust has been approximated with a polytropic EoS that is smoothly matched to those of the core and the outer crust, as in Ref. [6].

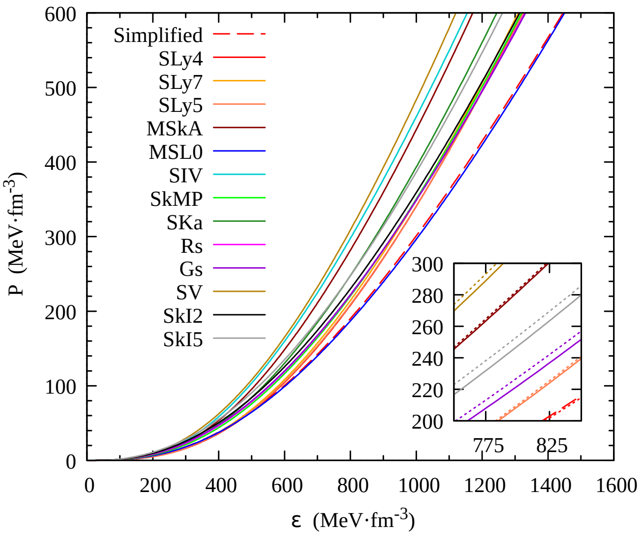

FIG. 1: EoS for all the Skyrme parametrizations used in this work. The simplified Skyrme EoS studied analytically has also been included. The inset shows the difference between accounting for the presence of muons (solid lines) and not doing so (dashed lines) for a few Skyrme interactions.

C. Tidal Deformability

When a spherical NS of mass M is subjected to the effect of a gravitational tidal field $\mathcal{E}_{ij}$, such as the gravitational influence between two neutron star in an early inspiralling binary system (to linear order), it develops as a response a quadrupole moment Q_{ij} [4,5].The parameter that relates $\mathcal{E}_{ij}$ and Q_{ij} is the tidal deformability (λ):

$$\lambda = -\frac{Q_{ij}}{\mathcal{E}_{ij}}. \tag{13}$$

The tidal deformability can be written in terms of the radius of the star and an adimensional parameter known as *Love number* (k_2) [4,5]:

$$\lambda = \frac{2}{3}\frac{k_2 R^5}{G}. \tag{14}$$

The calculation of the Love number requires for a differential equation to be solved altogether with the TOV equations [4,5]. As a consequence, k_2 is a parameter that depends on the whole energy-pressure profile inside the star, and thus it depends on the EoS used to describe it. Further detail on the calculation of the tidal Love number is given in the annex.

The adimensional tidal deformability (Λ) is defined:

$$\Lambda = \frac{2}{3}k_2\xi^{-5} = \frac{2}{3}k_2\left(\frac{Rc^2}{2GM}\right)^5, \tag{15}$$

where $\xi = 2GM/Rc^2$ is the dimensionless compactness parameter.

In a NS binary system, however, the tidal deformability of one of the stars is not a direct observable. The measurable quantity responsible for the total (corresponding to both stars) effect of the tidal deformability on the phase evolution of the GW signal is the mass-weighted tidal defformability ($\widetilde{\Lambda}$) of the binary system [3]:

$$\tilde{\Lambda} = \frac{16}{13}\frac{(M_1+12M_2)\,M_1^4\Lambda_1 + (M_2+12M_1)\,M_2^4\Lambda_2}{(M_1+M_2)^5}. \tag{16}$$

Another observable which can be extracted from the time evolution of the frequency in the observed GW signals is the chirp mass ($\mathcal{M}$):

$$\mathcal{M} = \frac{M_1^{3/5}M_2^{3/5}}{(M_1+M_2)^{1/5}}. \tag{17}$$

This magnitude can be constrained with much greater precision than the tidal deformability, and thus when working with the numerical calculations it can be fixed to be equal to an observed experimental value in order to compare the results obtained for $\widetilde{\Lambda}$ with the experimental data.

D. Moment of Inertia

The moment of inertia (I) for a NS, in the slow-rotation approximation ($\Omega \ll \sqrt{GM/R^3}$) is given by the following integral [6]:

$$I = \frac{8\pi}{3}\int_0^R r^4 e^{-\vartheta(r)}\frac{\bar{\omega}(r)}{\Omega}\frac{[\varepsilon(r)+p(r)]}{\sqrt{1-2Gm(r)/r}}dr. \tag{18}$$

where Ω stands for the angular velocity of the NS; and $\vartheta(r)$, $\bar{\omega}(r)$ are certain radial functions. Further information on the resolution of this integral, and on the calculation of the radial functions that appear in it, is given in the annex.
The moment of inertia is not a direct GW observable; however, as we shall see in section III,it can be related to the tidal deformability using existing universal relations involving I and Λ.

III. RESULTS

The TOV structure equations (1) and (2), together with those of the tidal deformability and the moment of inertia (see Annex), have been solved using a set of Skyrme EoS for the description of the NS core, which are: SLy4, SLy7, SLy5, MSkA, MSL0, SIV, SkMP, SKa, Gs, SV, SkI2, SkI5, and the simplified interaction presented in Eq. (7) −although this simplified EoS is only displayed in Fig. 2, and has not been used to obtain further results. Additionally, a selected number of diverse non-Skyrme EoSs has also been used in order to strengthen the generality of the derived relations. These

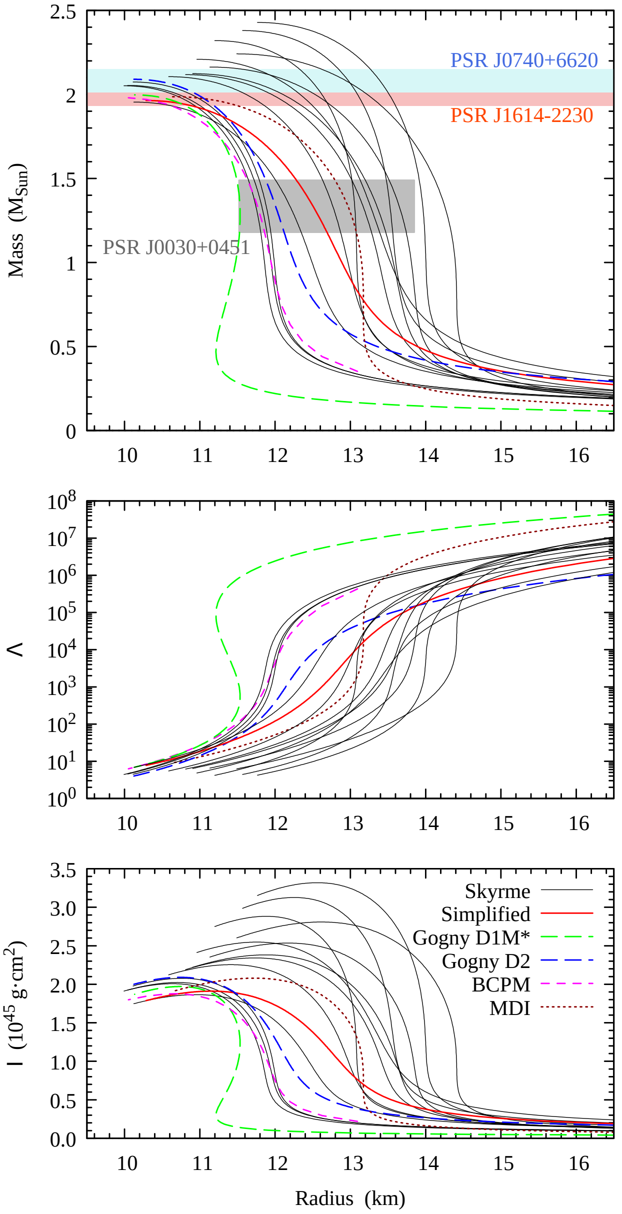

FIG. 2: Resulting M-R, $\Lambda-R$ and $I-R$ relations. In the M-R relation, we have added two recent observational constraints on the highest masses observed for NS, extracted from Refs. [9,10]); and another constraint on both M and R from recent NICER observations, Ref. [11].

additional EoSs are: Gogny D1M*, Gogny D2, BCPM, and MDI. The results are displayed in Fig. 2.

Inspired by the results obtained in Ref. [14], in Fig. 3 we have reproduced the so called *I-Love* relation, which stands as a universal relation between Λ and a renormalization of I, that is satisfied by all NS models with independence on the particular EoSs used. These two magnitudes (Λ and I) are usually not simultaneously measurable (e.g. GW detections can be used to constrain M and $\tilde{\Lambda}$ −from which restrictions for Λ can be obtained−;

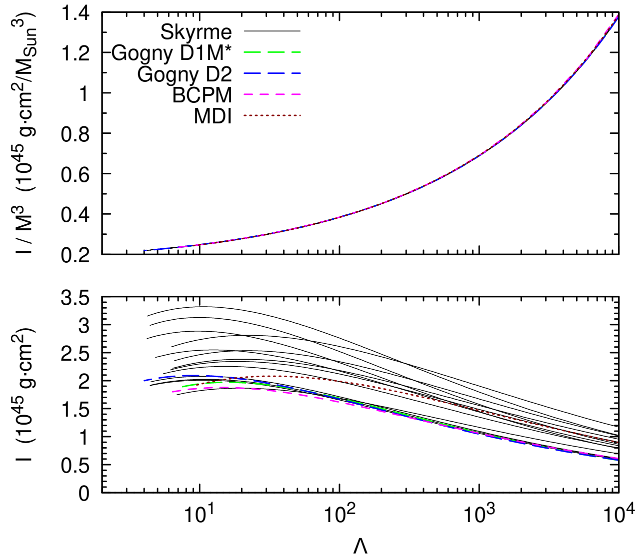

FIG. 3: Relation between a renormalization of the moment of inertia and the adimensional tidal deformability, the *I-Love* relation. Below, the direct relation between I and Λ.

but not I). Thus, being fulfilled without dependence on the nuclear model, this relation may prove useful in the derivation of experimental constraints on I, from Λ observations (or vice versa).

In Fig. 4, we have represented each model's prediction (i.e. the results of the calculations for each EoS) on R and Λ, for fixed values of M. As it can be seen, the resulting point spreads fall remarkably well within power laws. Therefore, as the results for all EoS tend to collapse on the same fits, these power laws can be considered as generic relations for NS; especially for increasing NS masses, were the point dispersion decreases. These relations may be useful both in terms of fireproofing EoSs (as those whose resulting NS models fail to satisfy this relations may be inappropiate to describe NS interiors), and in terms of obtaining constraints on either Λ or R from observations in which the other is measurable.

Nevertheless, in GW produced by coalescing NS binary systems, Λ is not a direct observable, and instead it is derived from the weighted average of Λ for both NS ($\widetilde{\Lambda}$), cf. Eq. (16). Because of this, in Fig. 5 we have implemented a representation analogous to that of Fig. 4 but representing, for each EoS, $\widetilde{\Lambda}$ and $R_{1.4}$ (radius of a *canonical* NS: $M = 1.4\ M_\odot$) at a fixed chirp mass corresponding to GW170817 ($\mathcal{M} = 1.186^{+0.001}_{-0.001}$ [12]). The point spread that results after doing so allows for a power law fit to be conducted. Furthermore, the same power law can be used for the point spreads of both values of the mass ratio ($M_1/M_2 = 0.7$ and 1). As such, this power law constitutes again a generic (although with finite dispersion) relation for NSs, that serves for multiple mass ratios (at the very least, for M_1/M_2 ranging between 0.7 and 1).

This power law does indeed allow for constraints on

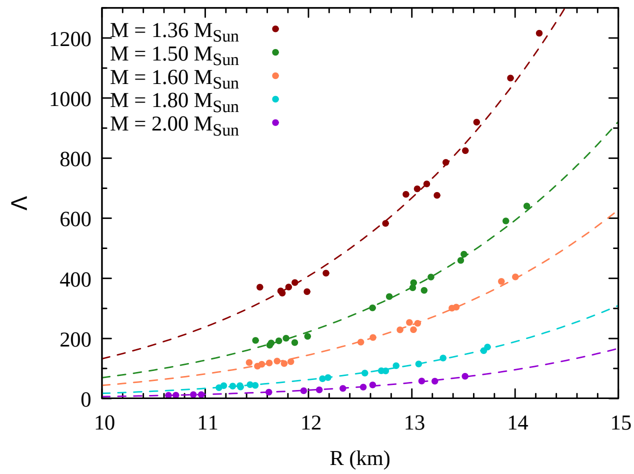

FIG. 4: Results for Λ and R at fixed values of M, for each EoS. $M = 1.36$ is equivalent to a symmetric ($M_1 = M_2$) binary system with the same chirp mass as the one observed in GW170817 [13].

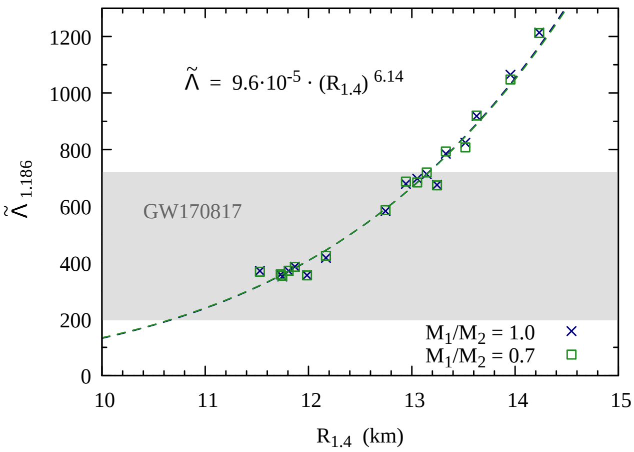

FIG. 5: Results for the mass-weighted $\widetilde{\Lambda}$ at fixed chirp mass ($\mathcal{M} = 1.186^{+0.001}_{-0.001}$ as in GW170817), and the radius of a star of $M = 1.4\ M_\odot$ ($R_{1.4}$); for each EoS. The observational constraints on $\widetilde{\Lambda}$ from GW170817 and AT2017gfo, as of [12,13], are: $\widetilde{\Lambda} \in (197, 720)$.

$R_{1.4}$ to be derived from direct observational data on $\widetilde{\Lambda}$. Being that so, using the power law fit given in Fig. 5, and using the restrictions on the mass-weighted tidal deformability extracted from the GW170817 event by the LIGO-Virgo collaboration [12] (and from its electromagnetic counterpart, AT2017gfo [13]), we have obtanied constraints on the radius of a canonical NS ($M = 1.4\ M_\odot$). The corresponding constraints are: $R_{1.4} \in (10.7, 13.2)$ km.

Furthermore, the given constraints on $R_{1.4}$ may also be used in relations like Fig. 4 to restrict the possible values of Λ and I for a canonical NS.

IV. CONCLUSIONS

- In this work we have performed the calculations for the structure, the tidal deformability and the moment of inertia for NS, and compared them graphically with some experimental restrictions.
- We have also reproduced the *I-Love* relation first found in [14].
- We have then deduced generic relations between Λ and R, for NS of different masses.
- Finally, we have deduced a generic relation between $\widetilde{\Lambda}_{1.186}$ of the NS binary system and $R_{1.4}$, valid for different values of M_1/M_2 (ranging between 1 and 0.7, at least); and we have used it, altogether with the observational constraints from the GW170817 event, to deduce constraints on the radius of a canonical NS: $R_{1.4} \in (10.7, 13.2)$ km.

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V. ANNEX

A. Tidal Love Number

All the following expressions have been extracted from Refs. [4,5], for further detail see [4,5] and references therein.

The tidal Love number is given by:

$$\begin{aligned} k_2(\xi, y_R) = & \frac{1}{20}\xi^5(1-\xi)^2\left[(2-y_R)+(y_R-1)\xi\right] \\ & \times \Bigg\{ \left[(6-3y_R)+\frac{3}{2}(5y_R-8)\xi\right]\xi \\ & + \frac{1}{2}\left[(13-11y_R)+\frac{1}{2}(3y_R-2)\xi+\frac{1}{2}(1+y_R)\xi^2\right]\xi^3 \\ & + 3\left[(2-y_R)+(y_R-1)\xi\right](1-\xi)^2\ln(1-\xi)\Bigg\}^{-1}, \end{aligned} \tag{19}$$

where $\xi = 2GM/Rc^2$ is the dimensionless compactness parameter. The quantity $y_R \equiv y(R)$ is another dimensionless parameter, solution of the following first-order differential equation:

$$\frac{dy(r)}{dr} = -\frac{y(r)^2}{r} - \frac{y(r)}{r}F(r) - rQ(r), \tag{20}$$

with boundary condition $y(0) = 2$, and:

$$F(r) = \left\{1 - 4\pi r^2\left[\varepsilon(r) - p(r)\right]\right\}\left[1 - \frac{2m(r)}{r}\right]^{-1}, \tag{21}$$

$$\begin{aligned} Q(r) = & 4\pi\left[5\varepsilon(r) + 9p(r) + \frac{\varepsilon(r)+p(r)}{\nu_s^2(r)} - \frac{6}{r^2}\right] \\ & \times \left[1 - \frac{2m(r)}{r}\right]^{-1} \\ & - \frac{4m^2(r)}{r^4}\left[1 + \frac{4\pi r^3 p(r)}{m(r)}\right]^2\left[1 - \frac{2m(r)}{r}\right]^{-2}, \end{aligned} \tag{22}$$

where $\nu_s^2(r) \equiv dp(r)/d\varepsilon(r)$ stands for the squared speed of sound. Equation (16) has to be solved together with the TOV structure equations, as it needs the functions $\varepsilon(r)$, $p(r)$ and $\nu_s(r)$. Thus, $y(r)$ is calculated from the center to the surface of the star.

There is an alternative method for the calculation of the Love number, which involves the solution of a system of two differential equations as described in Ref. [5]. Although it may be longer, as it involves multiple functions and two differential equations, as opposed to the method presented here, with only one differential equation corresponding to (20); both methods are indistinctively used in the literature. In our case, although we've presented the most compact method of the two, we have used the one corresponding to Ref. [5] in to compute k_2.

B. Moment of Inertia

The following expressions have been extracted from Ref. [6], for further information see [6] and references therein.

In the slow-rotation approximation ($\Omega \ll \sqrt{GM/R^3}$), the moment of inertia is given by Eq. (18). Under the same slow-rotation approximation, $\vartheta(r)$ can be determined from evaluating the following integral:

$$\vartheta(r) = \frac{1}{2}\ln\left(1 - \frac{2GM}{R}\right) - G\int_r^R \frac{\left[m(x) + 4\pi x^3 p(x)\right]}{x^2\left(1 - 2Gm(x)/x\right)}dx, \tag{23}$$

and $\bar{\omega}(r)$ can be obtained by solving the following differential equation on $\widetilde{\omega}(r) = \bar{\omega}(r)/\Omega$:

$$\begin{gathered} \frac{d}{dr}\left(r^4 j(r)\frac{d\widetilde{\omega}(r)}{dr}\right) + 4r^3\frac{dj(r)}{dr}\widetilde{\omega}(r) = 0, \\ j(r) = \begin{cases} e^{-\vartheta(r)}\sqrt{1 - 2Gm(r)/r} & \text{if} \quad r \le R \\ 1 & \text{if} \quad r > R, \end{cases} \end{gathered} \tag{24}$$

with the following two boundary conditions:

$$\widetilde{\omega}'(0) = 0 \tag{25a}$$

$$\widetilde{\omega}(R) + \frac{R}{3}\widetilde{\omega}'(R) = 1. \tag{25b}$$

In the slow-rotation approximation, the moment of inertia does not depend on Ω. Therefore, Eq. (24) can be integrated starting from an arbitrary value for $\widetilde{\omega}(0)$, up to the surface. Finally, a scaling of the function and its derivative (by a constant) will usually be needed in order to satisfy condition (25b); as arbitrary values for $\widetilde{\omega}(0)$ will not usually satisfy it.